How Math Can Be Taught Better

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Abstract

I have observed that exponents and logarithms are the stopping point for most failed high school math students. In this paper I will discuss how this subject can be taught better. The points I make apply to all topics in high school mathematics. For instance, I offer a geometric solution for carbon dating problems that provides two digits of accuracy and can be done in seconds, which is handy on timed multiple-choice tests. Also, I discuss what math is needed by tradesmen. That such practical men feel their high school math classes were irrelevant is one reason why so many dislike mathematics.

In the real world, this widespread dislike for mathematics is concealed by an almost religious belief in "data driven" research, which means statistics. But their mathiness is achieved by feeding empirical data into statistical software that they downloaded from the internet, not by actually doing any math. I have observed that the more strident a researcher is about proclaiming himself to be data driven, the more harshly he denounces deductivism, by which he means any type of mathematics other than statistics.

If hate for math were restricted to people who do not need it, then we could just say that math is not for everyone. But when even those who need it hate it, then we must look for systemic problems in how math is being taught. Thus, while this paper is primarily written for educators, it is of interest to everyone whose profession requires the use of mathematics.

Keywords

teaching math, math instruction, math education, exponents, logarithms, carbon dating

Section 1: Unneeded Intermediate Steps

There are often many ways to express the same information, each of which differ in how intuitive they seem to people with different backgrounds and with different motivations for considering the problem. For instance, computer programmers express the accuracy of a number in binary bits while nonprogrammers express accuracy in decimal digits. This is a rhetorical issue that can be resolved by identifying one's audience. Once the issue is understood to be rhetorical, it can be separated, perhaps brought to a section at the end of the chapter that is labeled as instruction on how to explain the material to nonprogrammers.

One multiplies the number of binary digits by $\log_{10} 2 \approx 0.30103$ because $\log_{10} 2^n = n \log_{10} 2$. If one is using signed integers on a 16-bit computer, there are $15 \times 0.3 = 4.5$ decimal digits of accuracy. This expression of accuracy is intuitive to people who are reporting their results in prose because it informs them that they must type four digits of each number. Another example of common logs being used for rhetorical purposes is Richter's scale, which news reporters use for expressing the magnitude of earthquakes when addressing the general public.

Indeed, the change of base formula, $\log_b x = \log_a x / \log_a b$, is almost always invoked due to rhetorical issues. All scientists everywhere use natural logarithms. Logarithms to other bases can only be described as contrived, appearing primarily so the author can demonstrate that he knows how to change bases. It has been 35 years since anyone has performed multiplication and division by consulting a table of common (base ten) logarithms, so they can be ignored.[1] Even in their day, they were expensive. Five-digit tables were beyond the means of students, yet machinists worked within this accuracy, as did loan officers calculating payments over $100.

The problem is that concepts like common logs are not often recognized as being rhetorical in nature. This leads an author to careen from one expression to another in the hopes that every reader will find at least something he has written to be intuitive. But the actual impression that they get of his writing is that he is making more work for himself and for his readers; that is, he is padding his book. The impression that math books are padded for length is the primary reason why students dislike the subject. My high school math teacher actually told us not to read the textbook, but I wore out backpacks carrying it because he assigned homework from it.

This impression is amplified by the tendency of textbook authors to take the euphemism "slow student" too literally. There is actually no such thing; "dull student" was a more descriptive

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[1] When I took algebra (1980), there was a giant slide rule mounted above the chalk board. The teacher said we would not be using it because, that year, calculators had made slide rules and logarithm tables obsolete; but for another year he would teach common logs because they were still on the SAT. Thirty-five years later, they still are.

term, though one we never hear anymore. Learning how logarithms work is like learning how to do a back flip. You either get it or you do not, but it is impossible to do it slowly. Being spoon-fed this information is boring, both for the bright students and for the dull students. The difference is that the former get past the boredom and learn the subject, usually unguided.

This too-literal interpretation of "slow student" is especially frustrating for students who are a week or two behind because they stumbled over something like logarithms. What they need is a succinct explanation that will bring them up to speed. If their textbook is unclear and they go to the book store, what they will find are remedial books with lots of happy "math is fun" talk and *very slow* explanations that run to hundreds of pages. The student lost a week of math instruction and now he is going to lose a lifetime of math instruction because that fat book might as well have "time to drop out" printed on the cover; this is the message he is getting.

Let us consider an example of an unneeded intermediate step. Money is often loaned for short periods with a fixed return. For example, I might loan you $100 and want $110 back next month. If you do not have the money, but I trust you, I might let you extend the debt another month. But now the loan is of $110, so you would owe me $121 the following month.

When comparing loans to each other, it makes sense to express all interest rates with the common unit of one year. This is a rhetorical technique because – in the same way that in the previous example we assumed the reader was reporting his results in prose when we converted to decimal digits – we are now assuming that there are several loans to be compared. If one just needs to know the proportional growth over time, then it is easier to raise one plus the rate to the number of recalculations, as $1.1^{18} \approx 5.56$ for this loan over eighteen months.

The debt described above is 10% every month or 120% every year. After eighteen months you would owe 5.56 times the original amount, called the principal[2], denoted P, which is $100.

$$\left(1 + \frac{r}{n}\right)^{nt} = \left(1 + \frac{1.2}{12}\right)^{12 \times 1.5} \approx 5.56 \qquad \text{The Standard Textbook Equation} \qquad (1)$$

It is easier to raise 1.1 to the 18th power than it is to figure out that the yearly rate is 120% when you are just going to divide this by 12 on the next step. The yearly interest rate, r, is an unneeded intermediate step. Students notice that they found r by multiplying by 12 and then, when evaluating their equation, the first thing they did was divide it by 12. But if they write $1.1^{18} \approx 5.56$, they get marked down because this is not the standard textbook equation.

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[2] Principle is a mental object, such as honesty. Principal is a physical object, such as the leader of a school or a sum of money. Note that principal describes money, not algae. The initial quantity of nonmonetary items is A or $f(0)$.

Section 2: Exponential Growth

When recalculating in discrete time intervals, the debt climbs like a staircase. What if we did these calculations continuously? Would the debt not grow in a smooth curve?

Let us first find the value of the money earned on 100% interest in one time period. If calculated once, 100% doubles the principal. Quarterly is $1.25 \times 1.25 \times 1.25 \times 1.25 \approx 2.44$ and continuously is $e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$, the limit of $\left(1 + \frac{1}{n}\right)^n$ as n grows without bound.

Raising a number to an integer power is easier if that integer is a power of two because then we can just square the number repeatedly. $\left(1 + \frac{1}{4}\right)^4 \approx 2.44$ is 1.25 squared twice because $2^2 = 4$. Since we are finding e, we cannot assume that we have a computer programmed to calculate an exponent, $x^y = e^{y \log x}$, but must find a technique that works on a four-function calculator.

Setting $n = 2^{18} = 262{,}144$ yields $e \approx 2.71828$. This is all the accuracy that high school students need. Scientists use $e \approx 2.718281828$, which looks like a repeating decimal, though the next four digits are not 1828 and it can be proven that e is irrational. This calculation was first done by Jacob Bernoulli in 1683, but it is called Euler's number because it was Euler who proved that it is irrational and, famously, that $e^{ix} = \cos x + i \sin x$, which is at the foundation of complex analysis, without which we would not have electrical engineering. Honor goes to mathematicians who prove theorems using deductive logic, not to people with the patience to do 18 multiplications, and certainly not to "data driven" researchers who just feed empirical data of dubious origin into statistical software that they downloaded from the internet.

Having found the value of the money earned on 100% interest in one time period, it is an easy step to allow the interest rate, r, to be something other than 100% and the time, t, to be something other than one time period.

$$p(t) = e^{rt} \qquad \qquad \text{Exponential Growth} \qquad \qquad (2)$$

p is a proportion – one may multiply both side by P, the principal – r is the interest rate and t is time. For our previous example, $e^{rt} = e^{0.1 \times 18} = e^{1.2 \times 1.5} \approx 6.05$. For the time unit defined as either a month or as a year, $rt = 0.1 \times 18 = 1.8$ or $rt = 1.2 \times 1.5 = 1.8$, the same thing.

Logarithms can now be defined as the inverse of exponents, with no mention of other bases.

Section 3: Exponential Decay

Recall the Law of Sines from trigonometry:

$$\frac{\sin \alpha}{a} = \frac{\sin \beta}{b} \qquad\qquad \text{Law of Sines} \qquad\qquad (3)$$

Here, a and b are sides of a triangle and α and β are the opposite angles. I have devised a law of similar appearance and proven in a similar manner describing exponential decay:

$$\frac{\log p_a}{t_a} = \frac{\log p_b}{t_b} \qquad\qquad \text{Law of Logs} \qquad\qquad (4)$$

Here, t_a and t_b are two different times in the future and p_a and p_b are the proportions remaining at those times. Equation (2) is called exponential decay when r is negative.

Proof:

$$p_a = e^{rt_a} \qquad\qquad\qquad \text{Equation (2)}$$

$$\frac{\log p_a}{t_a} = r \qquad\qquad\qquad \text{Log both sides and solve for } r.$$

$$\frac{\log p_a}{t_a} = r = \frac{\log p_b}{t_b} \qquad\qquad \text{Do the same for } p_b \text{ and } t_b; \text{ set equal.} \qquad \blacksquare$$

We can now eliminate the parameter r. This is similar to the proof of the Law of Sines where the equations for the sine of α and of β are both solved for height, which is then eliminated, leaving $\frac{\sin \alpha}{a} = \frac{\sin \beta}{b}$. Incidentally, just to keep things confusing, scientists call rate k instead of r. But we are eliminating rate from our equation, so we do not care what it is called.

Example: The half-life of carbon-14, a radioactive isotope of carbon, is 5730 years. This is the same thing as saying that its rate is -0.000121 or -0.0121% per year. Observe:

$$\frac{1}{2} \approx e^{-0.000121 \times 5730} \qquad\qquad \text{Carbon-14 with } p = \frac{1}{2} \qquad\qquad (5)$$

$\log \frac{1}{2} \approx -0.693$. Half-life is conventional, though, for a rhetorical reason, this convention was poorly chosen. The geometric interpretation (Section 5) defines decay in terms of tenth-life.

Half-life, not rate, is always given in problems about radioactive isotopes. This is because they vary far more than interest rates do. Bankers see 1% as a small rate and 20% as a financial crisis; but radioactive isotopes may take anywhere from seconds to millions of years to decay. Also, some are dangerous and half-life gives one an intuitive feel for when it will be safe again.

Discussing both half-life and rate in the same problem is another example of authors careening from one expression to another in the hopes that every reader will find at least something they have written to be intuitive. It is a rhetorical issue. Textbook authors should just decide if they are addressing scientists or bankers and then use the appropriate terminology.

$$\frac{\log p}{t} = \frac{-0.693}{T} \qquad\qquad \text{Law of Logs for Half-Life} \qquad\qquad (6)$$

We use the equals sign in (6) because it cannot be made more accurate than half-life, T, which is never known to more than three digits; $T = 5720$ makes tenth-life an even 19,000 years. Using $\approx$ would imply that (6) can be made arbitrarily accurate by citing $\log\frac{1}{2}$ to more digits.

$$\frac{\log p}{t} = \frac{-0.693}{5730} \qquad\qquad \text{Law of Logs for Half-Life of Carbon-14} \qquad (7)$$

Clearly, if you are given half-life, there are only two questions that can be posed on exams. Either they give you the time and ask of you the proportion remaining, or they give you the proportion remaining and ask of you the time to attain it. In both cases you cross multiply.

Example of being given the time and asked for the proportion:

> If the time since an organism died is 3310 years, what proportion of carbon-14 remains? Answer: Cross multiply (7) and then e the result, -0.4, to get 67%.

Example of being given the proportion and asked for the time:

> If the proportion of carbon-14 is 45%, how much time has passed since the organism died? Answer: Log 0.45 and then cross multiply to get 6600 years.

The half-life of iodine-131, used for detecting water leaks and treating thyroid cancer, as well as being released in weapons use, is 8.02 days. Long-term cancer risk is from breathing or ingesting caesium-137, whose half-life is 30.2 years. By cross multiplying the Law of Logs, we get the Century Law: *Every century after a nuclear attack reduces the radiation level tenfold.*

Section 4: Rule of 69

Calculating how long it takes for money at loan to double is of no practical consequence. Such questions are contrived for the purpose of associating the growth of money at interest with the decay of radioactive isotopes. Let us define d to be doubling time. (It is conventional for T and d to represent half-life and doubling time, though they can also be represented as $t_{0.5}$ and t_2.)

$$0.693 \approx rd \qquad\qquad \text{Rule of 69} \qquad\qquad (8)$$

Proof:

$$p(t) = e^{rt} \qquad\qquad \text{Equation (2)}$$

$$2 = e^{rd} \qquad\qquad \text{Consider doubling time.}$$

$$0.693 \approx rd \qquad\qquad \text{Log both sides.} \qquad\qquad ■$$

If the interest rate is 3%, it takes about 23.1 years to double one's money. In general, doubling time for 1% interest is about 69 years, and inversely proportional for other interest rates.

$$p(t) = 2^{t/d} \qquad\qquad \text{The Standard Textbook Equation} \qquad\qquad (9)$$

It is easy to prove that (9) is implied by our Law of Logs:

$$\frac{\log p}{t} = \frac{\log 2}{d} \qquad\qquad \text{Law of Logs for doubling time.}$$

$$\log p = \frac{t \log 2}{d} \qquad\qquad \text{Multiply by } t.$$

$$p = 2^{t/d} \qquad\qquad e \text{ both sides; recall that } x^{ab} = (x^a)^b. \qquad ■$$

To evaluate this equation, one must log it, take note that $\log x^y = y \log x$, and then e it. This is another example of introducing unneeded intermediate steps. Also note that it only works if one is given time and asked the proportion remaining; a whole slew of intermediate steps must be added to turn the equation around if one is given the proportion remaining and asked for time. The Law of Logs works equally well for both questions and is almost identical to the Law of Sines, both of which use cross multiplication that the student is already familiar with.

Following is a photocopy of a standard textbook, side-by-side with how I would teach this. Note that only money can grow exponentially; algae run out of food after a day or two.

Law of Logs Exposition	Traditional Exposition

If a function f satisfies (4), then we say that f **grows exponentially** if $k > 0$ and that f **decays exponentially** if $k < 0$. For future reference we notice that by the same analysis, if f satisfies (1) for all real numbers t, then f also satisfies (4) for all real numbers t.

Example 1 It is known that a certain kind of algae in the Dead Sea can double in population every 2 days. Assuming that the population of algae grows exponentially, beginning now with a population of 1,000,000, determine what the population will be after one week.

$$\frac{\log p_a}{t_a} = \frac{\log p_b}{t_b}$$ Law of Logs

Solution Let $f(t)$ denote the number of algae t days from now, for $t \geq 0$. We wish to find $f(7)$. By hypothesis, $f(0) = 1,000,000$, and combined with (4) this means that

$$f(t) = f(0)e^{kt} = 1,000,000e^{kt} \quad \text{for } t \geq 0 \qquad (5)$$

In order to find $f(7)$, we first determine k. By using (5) and the hypothesis that $f(2) = 2,000,000$ we deduce that

$$\frac{\log p}{7} = \frac{\log 2}{2}$$ Application

$$2,000,000 = f(2) = 1,000,000e^{2k}$$

Thus $\qquad\qquad 2 = e^{2k}$

so that by taking logarithms of both sides, we have

$p = 11.3$ Solve for p

$$\ln 2 = 2k$$

and therefore

11.3×10^6 algae in two days

$$k = \frac{1}{2}\ln 2$$

Consequently from (5),

$$f(t) = 1,000,000e^{(\ln 2)t/2} \qquad (6)$$

The right side of (6) can be simplified by noting that

$$e^{(\ln 2)t/2} = (e^{\ln 2})^{t/2} = 2^{t/2}$$

which means that

$$f(t) = 1,000,000(2^{t/2}) \quad \text{for } t \geq 0$$

To find $f(7)$, we use the last equation with $t = 7$ and find that

Only the accuracy you have!

$$f(7) = 1,000,000(2^{7/2}) \approx 11,313,709 \quad \square$$

The time it takes for a population to double is called its **doubling time**. Thus the doubling time for the algae in Example 1 is 2 days. More generally, if a population grows exponentially with doubling time d, then the solution of Example 1 can be modified to show that at time t the population is given by

This is equation (9).

$$f(t) = f(0)2^{t/d} \quad \text{for } t \geq 0$$

(see Exercise 6).

Section 5: Geometric Interpretation

Geometry students are taught how to construct the fourth proportional to three given line segments, though they are not sure why, since a calculator is more accurate. But, had the Law of Logs been invented before electronic calculators, constructions on poster-size semi-log paper would have been more accurate than a slide rule; a legal-size graph has two digits. Today, the motivation for the geometric solution for carbon-14 dating is speed on timed exams, assuming that a graph qualifies as a "calculator" under exam rules. Also, the geometric solution makes it clear why the decay of C-14 does not work for dating the remains of recently deceased people: The triangle is smaller and the percentages are much more closely spaced near the left vertex.

Law of Logs
Carbon-14 Dating
www.axiomaticeconomics.com/math.php
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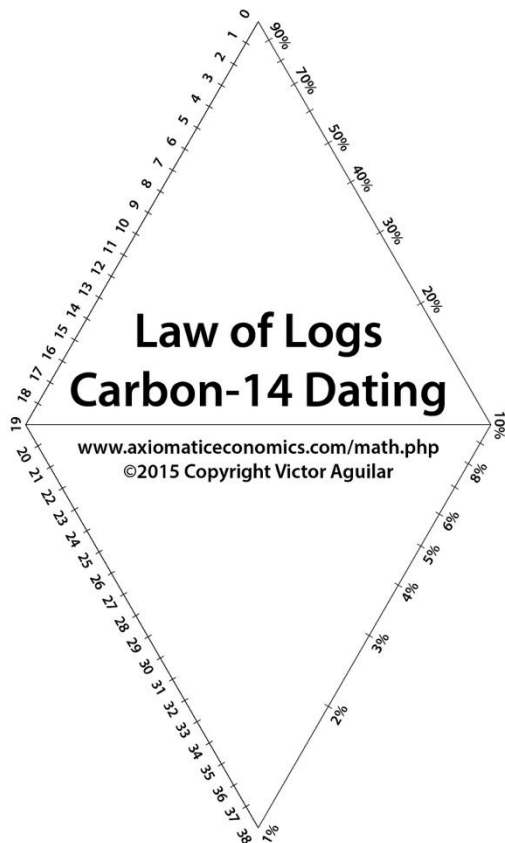
We define tenth-life to be exactly 19,000 years. By **Triangle Similarity** we know that

$$\frac{100\% \; to \; 50\%}{0 \; to \; 5720 \; years} = \frac{100\% \; to \; 10\%}{0 \; to \; 19,000 \; years}$$

This construction implies that $\frac{\log 0.5}{5720} = \frac{\log 0.1}{19,000}$

because the scale is logarithmic and $\log 1 = 0$.

This is a special case of $\frac{\log p_a}{t_a} = \frac{\log p_b}{t_b}$

Student exercise #1: Use the graph to find the proportion of carbon-14 remaining after 250 and 25,000 years. Estimate the accuracy of your answers. Does an eight-digit calculator answer both questions to the same eight digits?

Student exercise #2: Using semi-log paper, construct a 200-year geometric solver for caesium-137. Define tenth-life to be 100 years.

Modern geometry textbooks are silent on exponential decay, so we cannot compare their treatment of this problem to mine. Instead, I will illustrate (next page) how modern geometry textbooks are padded for length by overuse of common notions while omitting actual axioms, which results in long proofs with mincing steps. Common notions are axioms that, once established, can be skipped with a word: simplify. Axioms are what uniquely define a theory.

Given: $\overline{MR} \perp \overline{RP}$ and $\overline{QP} \perp \overline{RP}$; O is the midpoint of $\overline{RP}$

Prove: $\angle M \cong \angle Q$
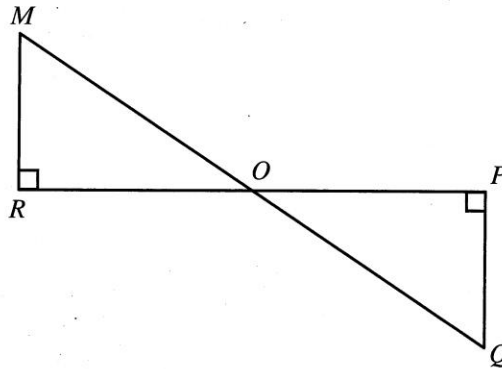
Statements	Reasons
1. $\overline{MR} \perp \overline{RP}$ and $\overline{QP} \perp \overline{RP}$	1. Given
2. $\angle MRO$ and $\angle QPO$ are right angles	2. Definition of perpendicular
3. $m\angle MRO = 90°$ $m\angle QPO = 90°$	3. Definition of right angles
4. $m\angle MRO = m\angle QPO$	4. Substitution property of equality
5. $\angle MRO \cong \angle QPO$	5. Definition of congruence
6. O is the midpoint of $\overline{RP}$	6. Given
7. $\overline{OR} \cong \overline{OP}$	7. Definition of midpoint
8. $\angle MOR \cong \angle QOP$	8. Vertical Angles Congruency Theorem
9. $\triangle MOR \cong \triangle QOP$	9. ASA Postulate
10. $\angle M \cong \angle Q$	10. CPCTC

Whenever segments are perpendicular, the intersection of these segments form right angles by the definition of *perpendicular.* These right angles measure 90° and are congruent. Using additional information from the given statement, $\overline{RO} \cong \overline{PO}$ because the midpoint of a segment separates the segment into two equal segments. Aside from the given information, the intersection of MQ and RP forms vertical angles. By the Vertical Angles Congruency Theorem, $\angle MOR \cong \angle QPO$. Because the triangles have two pairs of corresponding congruent angles and their included sides are congruent, the triangles are congruent by the ASA Postulate. CPCTC can then be used to prove any of the corresponding parts are congruent.

Proof requires Euclid's 4th postulate, that all right angles are equal; Euclid's 15th theorem, that opposite angles are equal; and Euclid's 32nd theorem, that the interior angles of a triangle sum to two right angles. Proof does *not* require that O be the midpoint of RP. But observe Bhatt's and Dayton's five mincing steps to prove that two right angles are equal, their needlessly restricting their theorem to congruent rather than just similar triangles, and their grounding all of their work on the unproven ASA "postulate," so nothing they did here has a solid foundation.

When asked if there were not easier means of learning geometry, Euclid famously replied that there is no royal road to geometry. But modern textbooks are written for the standardized exams. Bhatt and Dayton skipped over Euclid's five postulates and then declared his 26th theorem a postulate, probably because they had seen exam questions on the latter but not the former. Nor, apparently, do examiners ask about the sum of interior angles summing to 180°, and so it too was omitted; a grievous omission since it is equivalent to Euclid's fifth postulate.

Geometry today is just a jumble of statements randomly labeled as axioms (postulates) or as theorems (propositions) and never the same from one textbook to the next. Postulates are either missing entirely or they are just propositions that the author did not want to (or know how to) prove. Except for Euclid's first postulate, I find no mention of his other four postulates. The index of one textbook[3] indicates that "postulate" appears only once (p. 25), which I quote:

> *Both theorems and postulates are statements of geometric truth. The difference between postulates and theorems is that postulates are assumed to be true, but theorems must be proven to be true based on postulates and/or already-proven theorems. It's a fine distinction, and if I were you, I wouldn't sweat it.*

Mark Ryan does not make this distinction. I found a handful of Euclid's postulates, common notions and propositions scattered throughout and all are just stated as facts with no attempt to prove them. The student exercises have two-column "proofs" where the right-hand column consists of references to these facts/theorems/postulates, or whatever you want to call them.

Modern geometry classes might help you win on *Jeopardy*, but they will not help you become a scientist. Practical men do not want Alex Trebek for their teacher, but this is all that capital-E Educators are. Mark Ryan (p. 7) writes, "You'll have plenty of opportunities to use your knowledge about the geometry of shapes. What about geometry proofs? Not so much." The French core curriculum includes the ban, *teut expos'e de logique formalle est exclu* [any formal logic exposition is excluded] and Mark Ryan is just enforcing the American version of this ban.

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[3] Mark Ryan. 2011. **Geometry Essentials for Dummies**. Hoboken, NJ: Wiley Publishing Inc.

Section 6: Mathematics for Tradesmen

Not everybody is going on to college. One reason why exponents and logarithms are the stopping point for so many failed high school math students is because radioactive isotopes are seen as science, and students who are not going to be scientists feel that they need go no further. But people in all walks of life must be able to cross multiply to solve a proportion. Since the Law of Logs is a proportion, it is good practice for them to work with this equation.

The mathematics that is most relevant to the trades is the Pythagorean Theorem and the Quadratic Formula. Because the Pythagorean Theorem is a second-order equation and the Quadratic Formula solves second-order equations, they are to the tradesman what the one-two punch is to the pugilist. Let us consider some examples from several different trades:

1) A carpenter builds a rectangular cabin with inside dimensions ten feet by twenty feet. A two-foot diameter stove is in the center. The windows must be at least six feet from the stove lest the curtains catch fire. How far from the corners can the windows be built?

2) A trucker is to take a flatbed trailer eight and a half feet wide to the shipping yard, where he will pick up a corrugated steel pipe twelve feet in diameter. He must construct braces on the sides of his trailer to prevent the pipe from rolling off. How high should the braces be?

3) A plumber is to run a pipe through a hole six inches high and eight inches wide. A four-inch PVC pipe (o.d. 4.5") already goes through this hole. How large of a pipe will fit beside it?

4) A machinist cuts the upper right edge off of a 3.5" square bar with a 1.5" concave radius that leaves a 0.75" step on the lower right edge. He must now cut a convex radius on the upper left edge of the bar so it meets the concave radius and reduces the height of the bar to 2.75". What is the measure of this convex radius?

The first step to employing the Pythagorean Theorem is to construct a right triangle. In the first two problems, a circle overlaps a line, so drop a perpendicular from the center of the circle to the line and a hypotenuse to their intersection. In the last two problems, two circles touch, so the hypotenuse goes from center to center. Solutions: 1) 5' 1"   2) $21\frac{3}{16}$"   3) 3.9"   4) 1.028"

Practical problems such as these will hold the attention of students going into the trades. They will also hold the attention of college-bound students because bright college boys live in fear of being embarrassed by a "dumb" blue-collar worker when it comes to mathematics.