An Alternative Postulate Set for Geometry

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ABSTRACT

About 2300 years ago, Euclid wrote *The Elements*, which founded geometry on five postulates and some "common notions." There was much virtue in his system, but, by modern standards, its reliance on superposition for the proof of Proposition IV, SAS congruence, was not supported. Also, in light of recent developments in abstract algebra, Euclid's common notions are inadequate.

In 1898, David Hilbert published *Foundations of Geometry*, which omitted any mention of superposition and took SAS congruence as a postulate. As is to be expected of a mathematician coming two millennia later, Hilbert's foundations are a great improvement over Euclid's. However, SAS congruence is not very intuitive; it seems it should be grounded on more fundamental postulates that are intuitive. Also, Hilbert mixes geometric postulates and abstract algebra axioms together, which this author feels should be kept separate.

In 1932, George Birkhoff published *A Set of Postulates for Plane Geometry*, which were metric; that is, they assume that real numbers can be associated with any length, angle or area. This is assuming a lot. Once the triangle similarity theorem is accepted as a postulate (the transversal theorem is also a postulate), then every theorem is an easy corollary of these big assumptions. Also, while Birkhoff did not intend for this to happen, his followers often assign real numbers to lengths and angles, forget their meaning, and then add them together. There is no such thing as the sum of a length and an angle.

The purpose of this paper is to introduce a new set of postulates for geometry and to define the minimum of abstract algebra axioms needed. Emphasis is put on accepting as intuitive only those spatial relations that small children understand without explanation; their parents are just assigning names to concepts that are instinctive in humans. Geometers are invited to compare these assumptions with the foundations that are used in other textbooks. These postulates and axioms are to be the foundations of a textbook, Geometry–Do.

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Euclid's Postulates Plus One More

Segment	Two points fully define a segment.
Triangle	Three points fully define a triangle.
Line	A segment fully defines a line.
Circle	The center and the radius fully define a circle.
Right Angle	All right angles are equal to each other.
Parallel	A line and a point not on it fully define the parallel through that point.

The postulates are in terms of fully defined, which means that a figure with the given characteristics is unique, if it exists. SSS fully defines a triangle, but it may not exist if one side is too long; ASS does not fully define a triangle. Under defined means figures with the given characteristics are legion; more information is needed. John Playfair stated the parallel postulate roughly as I and David Hilbert do, which can be proven to be equivalent to Euclid's Fifth Postulate.

> *If a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.*

While Hilbert and I both found Euclid's postulate to be convoluted and chose Playfair's version, and we both reject real numbers as unsupported by our postulates, we otherwise took separate paths. *Geometry–Do* is like Hilbert's geometry, but it is unique and has its own postulates.

Euclid also had five "common notions," which vaguely describe what modern mathematicians call equivalence relations, total orderings and additive groups.

Equivalence Relations and Total Orderings

A relation is an operator, R, that returns either a "true" or a "false" when applied to an ordered pair of elements from a given set. For instance, if the set is integers and the relation is equality, then $5 = 5$ is true, but $5 = 4$ is false. Relations must be applied to objects from the same set. For instance, $\overline{EF} = \angle G$ is neither true nor false; it is incoherent. There are four ways that relations may be characterized. It is never true that a relation has all four, but some have three.

Transitive	$a\ R\ b$ and $b\ R\ c$ implies $a\ R\ c$
Reflexive	$a\ R\ a$
Symmetric	$a\ R\ b$ implies $b\ R\ a$
Anti-Symmetric	$a\ R\ b$ and $b\ R\ a$ implies $a = b$

A relation that is transitive, reflexive and symmetric is called an equivalence relation. The equivalence relations considered in geometry are equality, $=$, which applies to segments, angles or areas; congruence, $\cong$, which applies to triangles; similarity, $\sim$, which applies to triangles; and parallelism, $\parallel$, which applies to lines. $\overleftrightarrow{EF} \parallel \overleftrightarrow{GH}$ means that $\overleftrightarrow{EF}$ and $\overleftrightarrow{GH}$ do not intersect.

Since segments are known only by their length, $\overline{EF} = \overline{GH}$ means that $\overline{EF}$ and $\overline{GH}$ are the same length. It does not mean that they are the same segment; they may be in different locations. Since length is the same regardless of direction, it is always true that $\overline{EF} = \overline{FE}$. But triangles are known, not by just one magnitude, but by six. The vertices are ordered to show which ones are equal. $\overline{EFG} \cong \overline{HIJ}$ means that $\overline{EF} = \overline{HI}$, $\overline{FG} = \overline{IJ}$, $\overline{GE} = \overline{JH}$, $\angle E = \angle H$, $\angle F = \angle I$ and $\angle G = \angle J$. Beware! Writing the vertices of a triangle out of order is one of the most common mistakes made by beginning geometers, and it is always fatal to a proof.

A quadrilateral is a union of two triangles; congruence or similarity holds if and only if both pairs of triangles are congruent or similar. If $\overline{EFG} \cong \overline{IJK}$ and $\overline{EHG} \cong \overline{ILK}$, then, $\overline{EFGH} \cong \overline{IJKL}$. Analogously, if $\overline{EFG} \sim \overline{IJK}$ and $\overline{EHG} \sim \overline{ILK}$, then, $\overline{EFGH} \sim \overline{IJKL}$. Similarity is defined as two triangles with all corresponding angles equal, so $\overline{EFG} \sim \overline{IJK}$ and $\overline{EHG} \sim \overline{ILK}$ means that six pairs of corresponding angles are equal. This is more than just saying that the four corresponding interior angles of $\overline{EFGH}$ and $\overline{IJKL}$ are equal; thus, it is not true that proving these four equal is sufficient to prove $\overline{EFGH} \sim \overline{IJKL}$. A counter-example is a square and a rectangle; they both have all right angles, but they are not similar. This is one reason why we do not define quadrilaterals as four-sided figures. This is a vacuous definition that has led many beginners to err by claiming that a square and a rectangle are similar. Also, our definition makes quadrilaterals a continuation of triangles; American schools have these as semester programs that can be taken in either order.

A relation that is transitive, reflexive and symmetric is an equivalence relation and there are four in geometry: equality, congruence, similarity and parallelism. Relations that are anti-symmetric can only be defined if we have already defined equality, because equality is referenced in its definition. A relation that is transitive, reflexive and anti-symmetric is called an ordering. Geometers only consider one: less than or equal to, $\leq$. An ordering is total if $a \leq b$ or $b \leq a$, always. A set with both an equivalence relation, $=$, and a total ordering, $\leq$, is called a magnitude. There exist orderings that are not total, such as subset, but these are not used in geometry. Less than, $<$, means $\leq$ but not $=$. It cannot be defined until both $\leq$ and $=$ have been defined.

Note that our definition of magnitude does not imply that real numbers can be associated with lengths, angles or areas; only that the relations $=$ and $\leq$ exist and have the required properties. It does imply that magnitudes are unique, which is what the replication axiom below is stating.

Equal magnitudes are an equivalence relation and can be reproduced wherever needed; that is, compasses do not collapse when lifted from the paper but are like holding a rope at a length. Compasses that collapse would be like surveyors who can walk a rope around an arc but, the moment the center guy takes a step, their rope turns to smoke. Because errors accumulate, it is not possible to put hash marks every foot – a quarter-inch error in every mark is an error of several feet per hundred yards – plus shrinkage or expansion as temperature and humidity change. This is why we use a straight edge, not a ruler; but the idea that a compass cannot be lifted off the paper to mark a length elsewhere makes geometry a parlor game, not a science.

An equivalence class is a set of objects that are equal, congruent, similar or parallel to each other. Equivalence classes can be defined in reference to an existing equivalence class. For instance, if an equivalence class is defined as all the angles equal to a given angle, then all the angles complementary to any member of that class are equal to each other; that is, they form their own equivalence class. All the angles supplementary to any member of that class are also equal to each other. If an equivalence class is defined as all the lines parallel to a given line, then all the lines perpendicular to any member of that class are parallel to each other. All the circles with radii equal to any member of an equivalence class of equal segments are an equivalence class.

Equivalence also refers to statements that can be proven if the other one is assumed, and in either order. For instance, Euclid’s fifth postulate and Playfair’s postulate are equivalent because, assuming either to be true, it is possible to prove that the other is true. The equivalence of theorems can be expressed by separating them with the phrase “if and only if,” which can be abbreviated “iff.” Proof in the other direction is called the converse; that is, if X implies Y, then the converse is that Y implies X. If X and Y are equivalent, then both implications are true.

Proof by contradiction when there is only one alternative that must be proven impossible is called a dichotomy. A trichotomy (e.g. ASA congruence) has three alternatives. A magnitude can either be less than, equal to or greater than another, and only one of these three is desired; thus, by proving the other two to be impossible, we know that it is the one that makes the theorem true.

Additive Groups

We define an additive group as a set and an operation (addition) that has these properties:

Associative property	$(a+b)+c = a+(b+c)$
Commutative property	$a+b=b+a$
Existence and uniqueness of an identity	$a+0=a=0+a$
Existence of inverses (identity is its own)	$a+(-a) = 0 = (-a)+a$

There exist magnitudes that are not additive groups, such as economic value. Given a choice between a or b, it is always possible for a person to choose one above the other. But, because a may substitute for or be a complement to b, they are not independent the way geometric magnitudes are. There are also additive groups that cannot be ordered, such as matrices. Matrices of the same dimension are an additive group, but we cannot say $a \leq b$ for any two.

On the first day of class I ask the students to look back to a time eight or ten years prior, when they were little kids and knew only how to add and subtract; multiplication and division was still scary for them. I assure them that geometry will be like going back to 1st grade. Sticking segments together end to end or angles together side by side is no more difficult than 1st grade problems about adding chocolates to or subtracting chocolates from a bowl of candies. How easy is that?

Replication Axiom

Given $\overline{EF}$ and $\overrightarrow{E'G'}$, there exists a unique point F' on $\overrightarrow{E'G'}$ such that $\overline{EF} = \overline{E'F'}$.

Given $\angle EFG$ and $\overrightarrow{F'E'}$, there exists rays $\overrightarrow{F'G'}$ and $\overrightarrow{F'G''}$ such that $\angle EFG = \angle E'F'G' = \angle E'F'G''$.

Interior Segment Axiom

If G is between E and F, then $\overline{EG} < \overline{EF}$ and $\overline{GF} < \overline{EF}$ and $\overline{EG} + \overline{GF} = \overline{EF}$.

Interior Angle Axiom

If H is inside $\angle EFG$, then $\angle EFH < \angle EFG$ and $\angle HFG < \angle EFG$ and $\angle EFH + \angle HFG = \angle EFG$.

Pasch's Axiom

If a line passes between two vertices of a triangle and does not go through the other vertex, then it passes between it and one of the passed vertices.

To be between E and F means to be on the segment they define, $\overline{EF}$, but at neither endpoint. To be inside $\angle EFG$ means to be between points on $\overrightarrow{FE}$ and on $\overrightarrow{FG}$, with neither point being F. It is instinctive that all humans know what it means for a point to be between two points and – in the case of Pasch's axiom – also what it means for a segment to be continuous; that is, with no gaps where another segment might slip through. Triangles and quadrilaterals are defined to be convex; the segment between two points interior to two sides is inside the figure. This means that they are not allowed to be concave or degenerate. Interior angles are greater than zero and less than straight, so triangles are never segments and quadrilaterals are never triangles or darts.

In *Geometry–Do*, between, inside, plane, point, shortest path and straight are undefined terms. These are concepts that a parent does not have to explain to a child; they are just giving names to concepts that are already in the child's mind. Area is defined as the number of squares that fill a triangle or union of triangles. Like the ancient Greeks, we do not have a rigorous definition of limits but just rely on intuition; wheat plants are infinitesimal compared to fields, so weighing the wheat is almost like calculating a limit. Thus, area too is something that small children can understand without explanation. Defining area as the product of a rectangle's sides waits for *Volume Two: Geometry with Multiplication*. This definition of area is not intuitive to small children, who know nothing of multiplication, which is why we divide our work into two volumes.

Degrees of angle will not be defined in either volume because doing so is trigonometry.

Triangle Inequality Theorem
Any side of a triangle is shorter than the sum of the other two sides.

In ancient Greece, Epicurus scoffed at Euclid for proving a theorem that is evident even to an ass (donkey), who knows what the shortest path to a bale of hay is. Indeed, it is a direct result of our definition that a segment is all the points along the shortest path between two points. It is an exercise for yellow belts to prove it using the greater angle and greater side theorems, but we will satisfy both Epicurus and Euclid by introducing it among the axioms while calling it a theorem.

The foundations explained above are sufficient through blue-belt study. In these early chapters, students will learn to bisect, trisect and quadrisect a segment, and to multiply it by small natural numbers by using repeated addition. No more of these repeated additions are needed than four, for construction of the Egyptian or 3–4–5 right triangle, the only exception being that we mention in passing the 5–12–13 right triangle, which is used by plumbers when installing 22.5° elbows.

Beginners, especially construction workers anxious to complete white-belt geometry, are advised not to get too hung up on these foundations, which are a bit abstract. But it is essential that we lay a solid foundation for our science. It is recommended that students read again about foundations when they are orange belts and are more comfortable with abstract reasoning. (Also, SSS and ASS, mentioned in the first paragraph, will then be known to them.) By then, those who are not – the construction workers – will be gone. Red belts are expected to teach beginning students to relieve black belts of this task. Pedagogical instruction is provided to red belts for this purpose, and they are also asked to read this foundational material yet again, and deeply.

Black belts will learn of similarity and prove the triangle similarity theorem. Similarity opens up a whole new world in geometry! Specifically, black belts will go beyond bisecting and trisecting segments to constructing segments whose length relative to a given unit is any rational number. But, for this, another axiom is needed. We have said that a set with both an equivalence relation,

$=$, and a total ordering, $\leq$, is called a magnitude. But to construct segments whose length relative to a given unit is any rational number, length must also be Archimedean.

Archimedes' Axiom

Given any two segments $\overline{EF}$ and $\overline{GH}$, there exists a natural number, n, such that $n\overline{EF} > \overline{GH}$.

This may seem trivially true, but Galois (finite) fields are not Archimedean. Every school boy is taught that Archimedes claimed that, given a long enough lever and a fulcrum to rest it on, he could move the world. They typically receive no clear answer from their teacher on why it matters, since no such fulcrum exists and Archimedes seems to ignore that gravity is attractive. The point that Archimedes is making is that, if there were such a fulcrum and much gravity under it, he would need a lever $6{\times}10^{22}$ longer on his side of the fulcrum to balance his mass against the Earth. If the fulcrum were one meter from Earth, Archimedes would be in the Andromeda galaxy if he stood on the other end of that long lever. $6{\times}10^{22}$ is a big number, but it does exist.

We said above that undefined terms are concepts that one does not have to explain to a child; the adult is just giving names to concepts that are already in the child's mind. But defining natural numbers as $1, 2, 3, \cdots$ is only intuitive up to as many fingers as the child has. When I took my four-year-old to another town, she was surprised that a different man was driving the bus. She thought that the few dozen people she had met in our town represented everybody in the world; that is, she thought that the natural numbers are a Galois field modulo 47. We think $6{\times}10^{22}$ exists because countably infinite fields are consistent; but so are big Galois fields. This axiom is why it is traditional in America to tell children that every snowflake is unique. That Archimedes' axiom is not intuitive to small children is one reason why similarity is delayed until black belt.

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