Appendix A: Alternative Distributions for First-Unit Demand

In section 11.3 the question was raised: What if people had traditionally assumed that the time value of money is an harmonic function rather than an exponential function? The present value of money to be received a year from now is currently calculated by multiplying the amount by a constant near unity called the common ratio. Under the proposed system, one would invert, add a constant, and invert again. By the Law of Proportionate Effect (Theorem 1, section 11.3), when the common ratio is taken to be not a constant but a random variable, first-unit demand is logrithmico-normally distributed (abbreviated lognormal). It is natural to make a similar inquiry into the distribution of first-unit demand when the value of money progresses harmonically:

$$m_j = \frac{1}{\frac{1}{m_{j-1}} + \varepsilon_j}$$

Each step is the last one inverted, added to a random quantity, and inverted again.

$$\frac{m_{j-1} - m_j}{m_j m_{j-1}} = \varepsilon_j$$

Solve for ε_j.

$$\sum_{j=1}^{n} \frac{m_{j-1} - m_j}{m_j m_{j-1}} = \sum_{j=1}^{n} \varepsilon_j$$

Find the sum of all ε_j from the initiation of the process to its termination after n steps.

$$\int_{m_0}^{m_n} \frac{-dm}{m^2} = \sum_{j=1}^{n} \varepsilon_j$$

If each step is small, $m_j - m_{j-1}$ can be approximated by dm.

$$\frac{1}{m_n} - \frac{1}{m_0} = \sum_{j=1}^{n} \varepsilon_j$$ Integrate from m_0 to m_n.

$$\frac{1}{m_n} = \frac{1}{m_0} + \varepsilon_1 + \ldots + \varepsilon_n$$ Solve for $\frac{1}{m_n}$.

Here, and throughout this appendix, the random variables associated with each day's change in value, $\{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n\}$, are taken from an unspecified distribution, but one with a finite mean and a non-zero, finite variance. This assumption is included in the characteristics of proportionate effect, in the characteristics of harmonic effect (each day's value changes by inverting, adding a random variable, and inverting again), and in any other axiom considered for the genesis of first-unit demand.

As can be seen from the last step, the reciprocal of one's indifference point after the n'th day is a constant (the reciprocal of its initial quantity) with a large number of random and identically-distributed quantities accumulated onto it. Hence, after having lived through n days and having seen their point of indifference change by a small amount each day, consumers of their first unit are normally distributed with regard to the variable $\frac{1}{m}$ and, hence, are reciprico-normally distributed with regard to the variable m. This relation shall be known as the Law of Harmonic Effect.

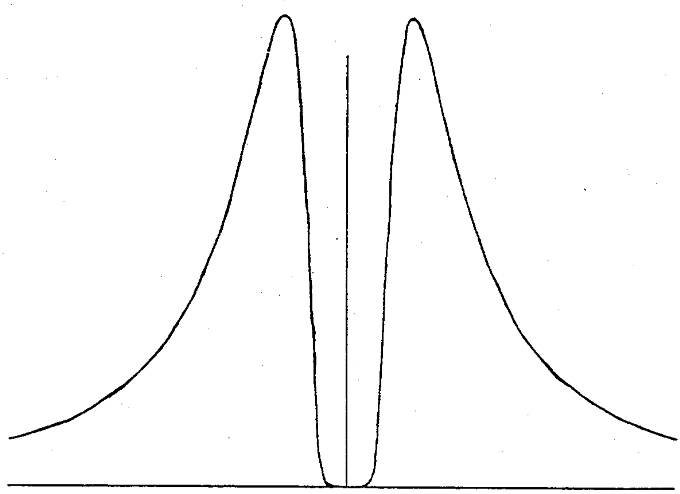
I will call reciproco-normally distributed phenomena ones whose reciprocals are normally distributed. The distribution is:

$$c_0(m) = \frac{e^{-\frac{1}{2}\left(\frac{\frac{1}{m}-\mu}{\sigma}\right)^2}}{\sqrt{2\pi\sigma m^2}}.$$ To save a division (divisions are costly in computer time), the linear transformation can be written $\frac{1-\mu m}{\sigma m}$. The reciproco-normal distribution is illustrated in figure 15.

change to Figure 16

Figure 15

The Reciprico-Normal Distribution



The properties of the arithmetic, geometric, and harmonic progressions are well known, though they are usually studied in isolation of one another. I propose that these are the first three of a sequence of progressions and, correspondingly, the normal, lognormal, and reciproco-normal distributions are the first three of a sequence of distributions.

Consider phenomena on which each day's events add a small quantity to that phenomena's value. If that quantity is always the same, it is called the common difference and the day-to-day values for that phenomenon follow an arithmetic progression. If that quantity is a random variable, in time the values of many such phenomena will be normally distributed. Notice that the quantity that is normally distributed, x, is an anti-derivative of 1, which may also be written x^0.

Consider phenomena on which each day's events multiply that phenomena's value by a quantity near unity. If that quantity is always the same, it is called the common ratio and the day-to-day values for that phenomenon follow a geometric progression. If that quantity is a random variable, in time the values of many such phenomena will be lognormally distributed. Notice that the quantity that is normally distributed, $\ln(x)$, is an anti-derivative of x^{-1}.

Consider phenomena on which each day's events invert that phenomena's value, add a small quantity, and invert again. If that quantity is always the same, the day-to-day values for that phenomenon follow an harmonic progression. If that quantity is a random variable, in time the values of many such phenomena will be reciproco-normally distributed. Notice that the quantity that is normally distributed, $\frac{1}{x}$, is an anti-derivative of $-x^{-2}$.

It is easy to see the pattern: Let an anti-derivative of x^0, x^{-1}, x^{-2},... be normally distributed and one has defined a sequence of distributions. The derivation of the associated sequence of progressions is as follows:

$$m_j = \frac{1}{\frac{1}{m_{j-1}} + m_{j-1}{}^k \varepsilon_j}$$

Each step is the last one inverted, added to a random quantity (which is multiplied by the k'th power of the last step), and inverted again.

$$\frac{m_{j-1} - m_j}{m_j m_{j-1}{}^{k+1}} = \varepsilon_j$$

Solve for ε_j.

$$\sum_{j=1}^{n} \frac{m_{j-1} - m_j}{m_j m_{j-1}{}^{k+1}} = \sum_{j=1}^{n} \varepsilon_j$$

Find the sum of all ε_j from the initiation of the process to its termination after n steps.

$$\int_{m_0}^{m_n} \frac{-dm}{m^{k+2}} = \sum_{j=1}^{n} \varepsilon_j$$

If each step is small, $m_j - m_{j-1}$ can be approximated by dm.

$$\frac{1}{(k+1)m_n^{k+1}} - \frac{1}{(k+1)m_0^{k+1}} = \sum_{j=1}^{n} \varepsilon_j$$

Integrate from m_0 to m_n.

$$\frac{1}{m_n^{k+1}} = \frac{1}{m_0^{k+1}} + (k+1)(\varepsilon_1 + \ldots + \varepsilon_n)$$

Solve for $\frac{1}{m_n}$.

As can be seen from the last step, the reciprocal of one's indifference point after the n'th day raised to the (k+1)'th power is a constant (the reciprocal of its initial quantity raised to the (k+1)'th power) with a large number of random and identically-distributed quantities accumulated onto it. Hence, after having lived through n days and having seen their point of indifference change by a small amount each day, consumers of their first unit are normally distributed with regard to the variable $\frac{1}{m^{k+1}}$. This relation shall be known as the Generalized Law of Harmonic Effect.

The generalized harmonic progression is given by $m_j = \frac{1}{\frac{1}{m_{j-1}} + m_{j-1}{}^{k}\varepsilon_j}$. This is the harmonic progression when k equals 0 and, for k = 1, 2, 3,..., one gets the latter progressions in the sequence. The distribution associated with the k'th progression is:

$$c_0(m) = \frac{(k+1)e^{-\frac{1}{2}\left(\frac{\frac{1}{m^{k+1}}-\mu}{\sigma}\right)^2}}{\sqrt{2\pi}\sigma m^{k+2}}$$

With the invention of any new distribution, a number of theorems must be proven regarding its mean, its variance, and a host of other characteristics used to describe statistical distributions. Also, an investigation must be made regarding what types of phenomena are modeled by the distribution. In the case of the reciprico-normal distribution, that means asking what types of phenomena change each day by inverting their current value, adding a random quantity, and inverting again. As I have invented not only the reciprico-normal distribution but an entire sequence of distributions, each with an associated progression, this is a fairly daunting task. However, the purpose of this appendix is not to introduce new research in pure mathematics, so the investigation of this sequence of distributions will be deferred for separate treatment. The purpose of this appendix is to attack my third axiom, that is, to attack my choice of the lognormal distribution for first-unit demand. Here, we will be content to address one application: Can the reciprico-normal distribution or a later distribution in the sequence be used to describe first-unit demand? If not, what steps must be taken to find an alternative to the lognormal distribution and are they executable?

The harmonic progression cannot be used to calculate the time value of money because its summation does not converge; that is, to replace our third axiom with the assertion that first-unit demand conforms to the characteristics of harmonic effect would imply that the present value of money held at interest is always infinite. Therefore, the reciproco-normal distribution is not a candidate for first-unit demand and, for the same reason, neither is the normal distribution. To test the latter distributions in the sequence, one must determine whether their associated progressions converge or diverge.

First, let us explicitly prove that the harmonic progression diverges: $f(t) = \dfrac{1}{c + \varepsilon t}$ diverges because $\displaystyle\int_1^\infty \frac{dt}{c + \varepsilon t} = \left.\frac{\ln(c + \varepsilon t)}{\varepsilon}\right|_1^\infty = \infty$ with c and ε positive constants. By the Integral Test for convergence, $\displaystyle\sum_{t=1}^{\infty} f(t) = \infty$. Now consider the value of this function one day after an arbitrary time, t. $f(t+1) = \dfrac{1}{c + \varepsilon(t+1)} = \dfrac{1}{c + \varepsilon t + \varepsilon} = \dfrac{1}{\dfrac{1}{f(t)} + \varepsilon}$ This last step is the harmonic progression from which the reciproco-normal distribution is derived. Thus, if $t = 0, 1, 2, \ldots$, then $f(t)$ parametrically defines the same ordered set of numbers that is defined recursively by the harmonic progression. Since $f(t)$ diverges, the harmonic progression diverges.

Notice that the smaller ε is, the faster $f(t)$ diverges. This is because, for every t, as ε tends toward zero the numerator of $\dfrac{\ln(c + \varepsilon t)}{\varepsilon}$ approaches $\ln(c)$, which is constant, and the denominator approaches zero.

Now consider the k'th progression in the sequence. Defined recursively, $f_k(t+1) = \dfrac{1}{\dfrac{1}{f_k(t)} + \varepsilon f_k(t)^k}$ with $\varepsilon > 0$. $f_k(t)$ is negative monotonic and it approaches zero as t approaches infinity, so there exists T such that, for all $t > T$, $f_k(t)^k < f_k(t) < 1$ and, hence, the k'th progression diverges faster than the harmonic progression. Thus, of the sequence of distributions beginning with the normal, lognormal, and reciprico-normal, only the lognormal qualifies for the distribution of first-unit demand because only the geometric progression's summation converges.

Let us explicitly list the steps that must be taken to provide an alternative to the axiom that first-unit demand conforms to the characteristics of proportionate effect, that is, an alternative to the lognormal distribution:

1) Choose a negative-monotonic function $f: \Re^+ \rightarrow \Re^+$ such that $\lim_{t \to \infty} f(t) = 0$.

2) Prove that the integral of $f(t)$ converges, that is, $\int_1^\infty f(t)dt < 0$.

3) Find the inverse (in composition), $f^{-1}(m)$, and set $m_j = f(f^{-1}(m_{j-1}) + \varepsilon_j)$. $f^{-1}(m)$ must be defined over all $\Re^+$.
4) Separate (3) so $g(m_j, m_{j-1})(m_j - m_{j-1}) = h(\varepsilon_j)$ with $h(\varepsilon_j)$ monotonic.
5) Anti-differentiate $g(m)$ to get $G(m)$ after making the approximations $m = m_j$ and $m = m_{j-1}$.
6) By applying the Central Limit Theorem, define the alternative distribution to be $c_0(m) = g(m)\dfrac{e^{-\frac{1}{2}\left(\frac{G(m)-\mu}{\sigma}\right)^2}}{\sqrt{2\pi}\sigma}$, scaled so that the area under it is unity.

Any of these steps may fail, in which case a new candidate must be chosen. Specifically, the integral of $f(t)$ may not converge, $f(t)$ may not be invertible, (3) may not be separable as described, or $g(m)$ may not have an anti-derivative. If a function meets all of these criteria except possibly (2), I will call the associated distribution "paranormal." Paranormal distributions for which the summation of the associated progression converges are demand distributions. It is an easy exercise to prove that stock is always finite regardless of which demand distribution is chosen.

First, let us consider the exponential function, $f(t) = e^{-t}$, to show how it produces the lognormal distribution:

1) $f(t) = e^{-t}$

2) $\int_1^{\infty} e^{-t} dt = -e^{-t}\Big|_1^{\infty} = e^{-1} < \infty$

3) $f^{-1}(m) = -\ln(m)$ with $m > 0$

Thus, $m_j = e^{-(-\ln(m_{j-1}) + \varepsilon_j)} = \dfrac{m_{j-1}}{e^{\varepsilon_j}}$

4) $m_j = \dfrac{m_{j-1}}{e^{\varepsilon_j}}$

$$m_j - m_{j-1} = m_{j-1}\left(e^{-\varepsilon_j} - 1\right)$$

$$\frac{m_j - m_{j-1}}{m_{j-1}} = e^{-\varepsilon_j} - 1$$

5) $g(m) = \frac{1}{m}$ so $G(m) = \ln(m)$

6) $$c_0(m) = \frac{e^{-\frac{1}{2}\left(\frac{\ln(m) - \mu}{\sigma}\right)^2}}{\sqrt{2\pi}\sigma m}$$

Now let us consider the p-series with p = 2. Two is a natural choice for p because $\sqrt{m_{j-1}} - \sqrt{m_j}$ has a conjugate.

1) $f(t) = \frac{1}{t^2}$

2) $$\int_1^\infty \frac{dt}{t^2} = -\frac{1}{t}\Big|_1^\infty = 1 < \infty$$

3) $f^{-1}(m) = \frac{1}{\sqrt{m}}$ with $m > 0$

Thus, $$m_j = \frac{1}{\left(\frac{1}{\sqrt{m_{j-1}}} + \varepsilon_j\right)^2}$$

4) $$m_j = \frac{1}{\left(\frac{1}{\sqrt{m_{j-1}}} + \varepsilon_j\right)^2}$$

$$\frac{\sqrt{m_{j-1}} - \sqrt{m_j}}{\sqrt{m_j m_{j-1}}} = \varepsilon_j$$

$$\frac{\sqrt{m_{j-1}} - \sqrt{m_j}}{m_{j-1}\sqrt{m_j} + m_j\sqrt{m_{j-1}}} = \varepsilon_j$$

5) $$g(m) = \frac{-m^{-\frac{3}{2}}}{2} \qquad \text{so } G(m) = \frac{1}{\sqrt{m}}$$

6) $$c_0(m) = \frac{e^{-\frac{1}{2}\left(\frac{\frac{1}{\sqrt{m}} - \mu}{\sigma}\right)^2}}{\sqrt{2\pi\sigma m^{\frac{3}{2}}}}$$

This proves that there is an alternative to the lognormal distribution for first-unit demand. I will call it the root-reciprico-normal distribution. It is similar in appearance to the lognormal distribution but it converges much more slowly. By converges more slowly I mean that, regardless of their respective linear transformations, the root-reciprico-normal distribution eventually dominates the lognormal distribution. By way of comparison, without any linear transformation ($\mu = 0$ and $\sigma = 1$) 99% of the probability in the lognormal distribution is from 0 to 10, while one must integrate the root-reciprico-normal distribution from 0 to 6400 to get 99% of its probability.

Hence, an entirely independent theory of economics could be constructed on the basis of the following three axioms:

1) One's value scale is totally (linearly) ordered:

- **i)** Transitive; $p \leq q$ and $q \leq r$ imply $p \leq r$
- **ii)** Reflexive; $p \leq p$
- **iii)** Antisymetric; $p \leq q$ and $q \leq p$ imply $p = q$
- **iv)** Total; $p \leq q$ or $q \leq p$

2) Marginal (diminishing) utility, u(*s*), is such that:

- **i)** It is independent of first-unit demand.
- **ii)** It is negative monotonic; that is, $u'(s) < 0$.
- **iii)** The integral of u(*s*) from zero to infinity is finite.

3) First-unit demand ~~conforms to proportionate effect:~~ is such that:

- **i)** Value changes each day by taking the square root of the previous day's value, inverting, adding a quantity (called ε_j, with j denoting the day), squaring, and inverting again.

ii) In the long run, the ε_j's may be considered random as they are not directly related to each other nor are they uniquely a function of value.

iii) The ε_j's are taken from an unspecified distribution with a finite mean and a non-zero, finite variance.

My choice of proportionate effect for first-unit demand (and, hence, my use of the lognormal distribution) has a solid intuitive basis because it is the generalization of calculating the interest owed on a debt per unit of time as a percentage of the amount owed (see section 11.3). Since this is how people have calculated interest throughout recorded history, the weight of tradition supports my choice of the lognormal distribution. It is unlikely that critics will succeed in convincing the banking industry to calculate the interest owed on a debt per unit of time by taking the square root of the amount owed, inverting, subtracting a constant, squaring, and inverting again. However, it is interesting that alternative systems exist which contain no internal contradictions but which are still very different from human economies. Perhaps the inhabitants of another planet organize their economy in this or another way, and consider their calculation of interest to be as natural as we consider multiplication by a constant.

In the proposed system, the value of money held at interest would grow considerably faster than exponentially. From step four (above), $m_j - m_{j-1} = m_{j-1}\left(e^{-\varepsilon_j} - 1\right)$ implies that the rate of growth for money held at interest is $\frac{dm}{dt} = km$ when first unit demand is modeled by the lognormal distribution. $\sqrt{m_{j-1}} - \sqrt{m_j} = \left(m_{j-1}\sqrt{m_j} + m_j\sqrt{m_{j-1}}\right)\varepsilon_j$ implies that the rate of growth for money held at interest is $\frac{dm}{dt} = -2km^{\frac{3}{2}}$ when first unit demand is modeled by the root-reciprico-normal distribution. Here k must be a negative constant to model savings rather than depreciation, which is why I said "subtract a constant" in the previous paragraph. This latter differential equation is separable and has a solution, $m = k^{-2}t^{-2}$, but that is not the solution we want. I do not know an explicit formula for accumulated compound interest under the proposed system. Because it dominates exponential growth, one cannot use a polynomial approximation. One might consider using the sequence e_0, e_1, e_2,... that I developed in section 12.1 or some similar method. I leave the approximation of a positive monotonic function $m(t)$ such that $\frac{dm}{dt} = -2km^{\frac{3}{2}}$ for $k < 0$ as an exercise for the reader.

If anything, critics of our system (exponential growth of money held at interest) should look for a function that meets all of the criteria listed above but for which the value of money held at interest grows less than exponentially. I am not aware that anyone has ever criticized the technique of multiplying by a constant or proposed any other formula, but some have been shocked by how much interest can accumulate over long periods of time. If a boy deposits some money in a bank and then returns for it as an old man, the bank will probably conjure up some rule about not paying off dormant accounts. This is partly because they can but also because the bank feels that the time value of money is accurately modeled by the exponential function for only a short time. People who are preparing to purchase a house with a 30-year mortgage are told how much their monthly payments will be but never the total amount of money that they will spend over the next 30 years. Most people would be appalled at the huge sum of money that they are going to spend relative to the cash price of their house. Thus, the proposed system, which dominates exponential growth, is not a practical alternative to exponential growth for money held at interest and, concomitantly, the root-reciprico-normal distribution is not a practical alternative to the lognormal distribution. I leave it as an exercise for the reader to find a practical alternative to the lognormal distribution. I would be interested in hearing from any of my readers who can successfully attack my third axiom or, for that matter, any of my axioms.